NONCOMMUTATIVE HARMONIC OSCILLATOR, ENTROPY AND ENTANGLEMENT

OSCILADOR ARMÓNICO NO CONMUTATIVO, ENTROPÍA Y ENREDO

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ABSTRACT. We show that the time evolution of the two dimensional noncommutative harmonic oscillator gives rise to entanglement. This is achieved at the level of the evolution operator by using the algebraic approach. The measure of entropy and the states of maximum entanglement are also considered in this context.

KEYWORDS: Time evolution, Hamiltonian, commutator, commutation relations, ladder operator.

RESUMEN. Demostramos que la evolución temporal del oscilador armónico dos-dimensional no conmutativo, da origen a estados enredados. Esto es logrado a nivel del operador de evolución utilizando una aproximación algebraica. La medida de la entropía y los estados de máximo enredamiento también son considerados en éste contexto.

PALABRAS CLAVE: Time evolution, Hamiltonian, commutator, commutation relations, ladder operator, harmonic oscillator, noncommutative.

Introduction

Noncommutative (NC) quantum mechanics is a laboratory where the properties of NC field theories can be studied. The inclusion of NC can be achieved in two different ways: using the star product on the space of ordinary functions, or defining the theory on a space that is intrinsically NC [1, 2]. In fact, the equivalence between the two approaches has been described in [3]. It has been shown that with the help of the associative star operation the study of NC field theories can be mapped into that of ordinary field theories [4, 5]. Generalizations of the Maxwell and Yang Mills systems have been considered [4, 5]. On the other hand, the theory of quantum information [6, 7] based upon the concept of quantum entropy, introduced firstly by Von Neumann [8], provides the required generalization of the notion of information [9]. The characterization of independent, correlated and entangled two qubit systems can be done in terms of conditional and mutual entropies [10]. For example, a maximally entangled two qubit system possesses negative conditional entropies and excessive mutual entropy. While a maximally correlated two qubit system possesses zero conditional entropies and maximal mutual entropy. It is worth remembering that for a bipartite system that starts in a pure state, the partial entropies of the two subsystems are equal at all times. This result follows from the equality, according to the Schmidt decomposition, between the eigenvalues of the partial density matrices. The time independent two dimensional NC harmonic oscillator has been considered by a number of authors [11], in the presence of a magnetic field (Landau problem) [12, 13], using a general linear transformation between NC and canonical coordinates [14] and even an effort has been done to generate entanglement of the resulting states [15]. In this paper, we consider the time evolution

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of the NC harmonic oscillator and show that entanglement is an inevitable consequence of NC. For this purpose, we calculate the evolution of the number state and the entropy for bipartite states containing k excitations. We conclude the section analyzing the entanglement as a direct consequence of the system entropies and obtain the maximal entanglement conditions.

Time evolution of the harmonic oscillator

The two dimensional harmonic oscillator is described by the Hamiltonian

$$H = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2\right) + \frac{m\omega^2}{2}\left(\hat{x}^2 + \hat{y}^2\right), \tag{1}$$

together with the commutation relations:

$$\begin{aligned} &[\hat{x}, \hat{y}] = i\hbar\theta, \\ &[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar, \\ &[\hat{p}_x, \hat{p}_y] = [\hat{y}, \hat{p}_x] = [\hat{x}, \hat{p}_y] = 0, \end{aligned} \tag{2}$$

where θ measures the noncommutativity. According to [16 -18] the physical system described by (1) and (2) is equivalent to the usual harmonic oscillator described by the Hamiltonian:

$$H = \frac{1}{2m}\left(1 + \frac{m^2\omega^2\theta^2}{4}\right)\left(p_x^2 + p_y^2\right) + \frac{m\omega^2}{2}\left(x^2 + y^2\right) + \frac{m\theta\omega^2}{2}\left(yp_x - xp_y\right) \tag{3}$$

with the conventional commutation relations. Now the creation and annihilation operators are defined as follows

$$\begin{aligned} a &= \sqrt{\frac{m\varpi}{2\hbar}}\left(x + i\frac{p_x}{m\varpi}\right),\ b = \sqrt{\frac{m\varpi}{2\hbar}}\left(y + i\frac{p_y}{m\varpi}\right), \\ a^{y} &= \sqrt{\frac{m\varpi}{2\hbar}}\left(x - i\frac{p_x}{m\varpi}\right),\ b^{y} = \sqrt{\frac{m\varpi}{2\hbar}}\left(y - i\frac{p_y}{m\varpi}\right), \end{aligned} \tag{4}$$

where

$$\varpi^2 = \frac{\omega^2}{1 + \frac{m^2\omega^2\theta^2}{4}}. \tag{5}$$

The commutation relations of the a, b and a^{y}, b^{y} are the usual for two independent quantum harmonic oscillators, *i.e.*:

$$[a, a^{y}] = [b, b^{y}] = I, \tag{6}$$

with the remaining commutators vanishing. So that the Hamiltonian (3) reduces to:

$$H = \Omega\left(a^{\dagger}a + b^{\dagger}b + 1\right) + ig\left(a^{\dagger}b - b^{\dagger}a\right) \tag{7}$$

where

$$\Omega = \hbar\varpi\left(1 + \frac{m^2\omega^2\theta^2}{4}\right),\ g = \frac{\hbar\omega^2\theta}{2}. \tag{8}$$

The number of excitations of the type a and type b oscillators are described by the operators $\hat{n}_a = a^{\dagger}a$ and $\hat{n}_b = b^{\dagger}b$ respectively. None of these operators commute with the Hamiltonian, and they are not conserved quantities. For that reason, it is convenient to introduce the observables:

$$\begin{aligned} \hat{N} &= \frac{a^{\dagger}a + b^{\dagger}b}{2} \\ Y_0 &= \frac{a^{\dagger}a - b^{\dagger}b}{2}. \end{aligned} \tag{9}$$

The former operator commutes with the Hamiltonian (7), *i.e.* it is an integral of motion, which is interpreted as the total number of excitations of the system. In atomic systems the latter operator Y_0 is known as *inversion of population* [19]. In the present context, the value of $\langle Y_0 \rangle$ measures the difference in the number of excitations of the two oscillators.

Whilst the time independent equation for the harmonic oscillator has been analyzed by a number of authors [11-15], as far as we know the time evolution has not been considered. In order to study the time evolution it proofs convenient to express the Hamiltonian in terms of the operators Y_0 already introduced and $Y_+ = a^{\dagger}b,\ Y_- = \left(Y_+\right)^{\dagger} = b^{\dagger}a$. These operators provide a representation for the $SU(2)$ algebra. Obviously the Hamiltonian, can be expressed as a linear combination of $\hat{N}, Y_+$ and Y_-:

$$H = \Omega\left(2\hat{N} + 1\right) + ig\left(Y_+ - Y_-\right), \tag{10}$$

while the time evolution operator is given by:

$$U(t) = \exp\left\{-i\frac{t}{\hbar}\left[\Omega\left(2\hat{N} + 1\right) + ig\left(Y_+ - Y_-\right)\right]\right\}. \tag{11}$$

Considering the following factorization (see for example [19, 20]):

$$\exp\left[\alpha_0 Y_0 + \alpha_+ Y_+ + \alpha_- Y_-\right] = \exp\left(\beta_+ Y_+\right)\exp\left(\ln\beta_0\, Y_0\right)\exp\left(\beta_- Y_-\right),$$

where

$$\beta_0 = \left(\cosh\Theta - \frac{\alpha_0}{2\Theta}\sinh\Theta\right)^{-2},\ \beta_\pm = \frac{2\alpha_\pm \sinh\Theta}{2\Theta\cosh\Theta - \alpha_0\sinh\Theta},\ \Theta^2 = \frac{\alpha_0^2}{4} + \alpha_+\alpha_-,$$

the time evolution operator $U(t)$ can be written in terms of the $SU(2)$ generators and $\hat{N}$ in such a way that each factor contains an operator or a linear combination of commuting operators:

$$U(t) = \exp\left[-i\frac{\Omega}{g}\tau\left(a^\dagger a + b^\dagger b + 1\right)\right]\exp\left(\tan\tau\, a^\dagger b\right)\exp\left[-\ln(\cos\tau)a^\dagger a\right]\exp\left[\ln(\cos\tau)b^\dagger b\right]\exp\left(-\tan\tau\, b^\dagger a\right) \quad (12)$$

with $\tau = gt/\hbar$. This result shows that the time evolution of the two dimensional harmonic oscillator leads to entanglement since the factorization necessarily involves operators containing simultaneously *a* and *b*, *i.e.* the evolution operator cannot be factored in terms of *a* dependent and *b* dependent operators solely. Before concluding this section let us consider the state $|n,m\rangle = |n\rangle_a \otimes |m\rangle_b$, where $|n\rangle_a$ and $|m\rangle_b$ are the number states for each oscillating mode. The total number of excitations is $k = n + m$. Obviously the state $|0,0\rangle$ is invariant, up to a phase, under the evolution (12) *i.e.*:

$$U(t)|0,0\rangle = \exp\left(-i\frac{\Omega}{g}\tau\right)|0,0\rangle. \quad (13)$$

Let us now consider a system whose initial state is $|0,k\rangle$, then we have:

$$|\Psi_k(t)\rangle = U(t)|0,k\rangle = \exp\left(-i\frac{\Omega}{g}\tau(k+1)\right)\cos^k\tau\sum_{n=0}^{k}\binom{k}{n}^{1/2}\tan^n\tau|n,k-n\rangle. \quad (14)$$

Notice that in this case the time evolution of a separable state leads to an entangled state and this is true only in the noncommutative case. In the commutative space case $(g = 0)$, the separability of the $|0,k\rangle$ is preserved by the time evolution. More about entanglement and entropy will be considered in the following section. The state $|\Psi_k(t)\rangle$ can be used to calculate the expectation values of $\hat{N}$ and Y_0. As the former is an integral of motion, its expectation value is constant in time $\langle\hat{N}\rangle = k/2$, while the latter turns out to be periodic in time:

$$\langle Y_0(t)\rangle = -\frac{k}{2}\cos 2\tau\,. \quad (15)$$

A discussion of the physical meaning and relevance of Y_0 in the context of two level atoms can be found in [21]. Recall that here the value of $\langle Y_0\rangle$ means the difference in the number of excitations between the oscillating modes *a* and *b*. Figure 1 shows the inversion of population $\langle Y_0\rangle$ for $k = 1, 2$ and 3 excitations. Absence of collapses and revivals is due to periodic energy transference between the modes of the oscillator. Another interesting physical feature is the Landau problem, that it has been solved by Bellucci *et* *al* [13] in the same frame of noncommutative coordinates and also momenta. They consider the magnetic field induces noncommutativity in momenta. They found that their Hamiltonian presents $SU(2)$ or $SU(1,1)$ symmetries depending of the magnetic field value. This means that entanglement also arises, due to intermediation of magnetic field. This phenomenon occurs in other physical systems, such as bipartite systems entangled by a thermal bath (see for example [22, 23]), or particles in a cavity immersed in a thermal bath (see for example [24, 25]).

Entanglement and entropies

The density matrix of the bipartite system is constructed in terms of the time dependent state (14). This is so because, even if the time evolution is given by an unitary operator, it is a non local one:

$$|\Psi_k(t)\rangle\langle\Psi_k(t)| = \cos^{2k}\tau \sum_{n,m=0}^{k} \binom{k}{n}^{1/2} \binom{k}{m}^{1/2} \tan^{n+m}\tau |n,k-n\rangle\langle k-m,m|. \tag{16}$$

The reduced density matrix corresponding to the a and b modes are given by:

$$\hat{\rho}_a = \cos^{2k}\tau \sum_{n=0}^{k} \binom{k}{n} \tan^{2n}\tau |n\rangle_a \langle n|_a , \tag{17}$$

$$\hat{\rho}_b = \cos^{2k}\tau \sum_{n=0}^{k} \binom{k}{n} \tan^{2n}\tau |k-n\rangle_a \langle k-n|_a . \tag{18}$$

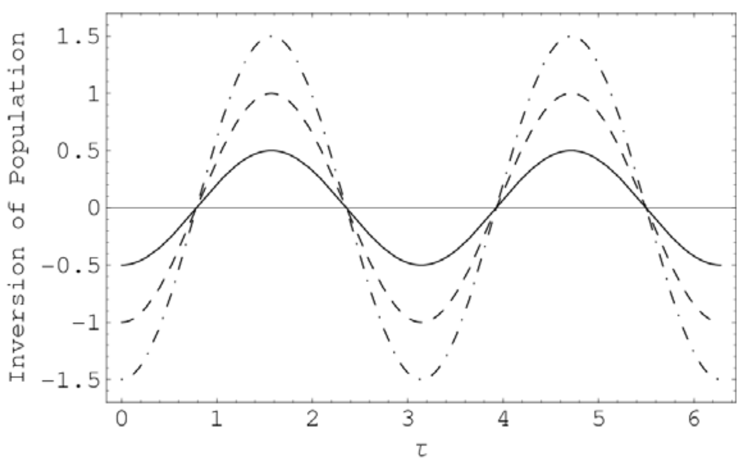

Figure 1. Inversion of population for k = 1 (continuous line), k = 2 (dashed line) and k = 3 (dot-dashed line).

Notice that these matrices are diagonal. The joint Von Neumann entropy S_{ab} is calculated in terms of the eigenvalues of the density matrix (16). Such eigenvalues are the same that those of the initial state, and therefore, it vanishes $S_{ab} = 0$. On the other hand, the partial entropies are equal in both subsystems:

$$S_a = S_b = -\sum_{i=0}^{k} p_i \ln p_i , \tag{19}$$

with

$$p_i = \binom{k}{i} \cos^{2k}\tau \tan^{2i}\tau\,, \tag{20}$$

while the conditional and mutual entropies[2] are

$$S(a|b) = S_{ab} - S_b = -S_b\,, \tag{21}$$

$$S(a:b) = S_a + S_b - S_{ab} = 2S_b\,. \tag{22}$$

Figure 2 shows the partial entropy for the particular cases k = 1, 2, 3. We can see that the partial entropy amplitude is positive and increases as the number of excitations increases. This permit us to conclude that the conditional entropy is negative, which is a signature for entanglement. Moreover, it has been shown [27] that if the conditional entropy is non negative, then the entropic Bell inequalities are fulfilled. The negative conditional entropy for the system under consideration implies violation of those inequalities. Note that besides the commutative case $\theta = g = \tau = 0$, the system is separable only when $\tau = gt/\hbar$ is an integer multiple of $\pi/2$.

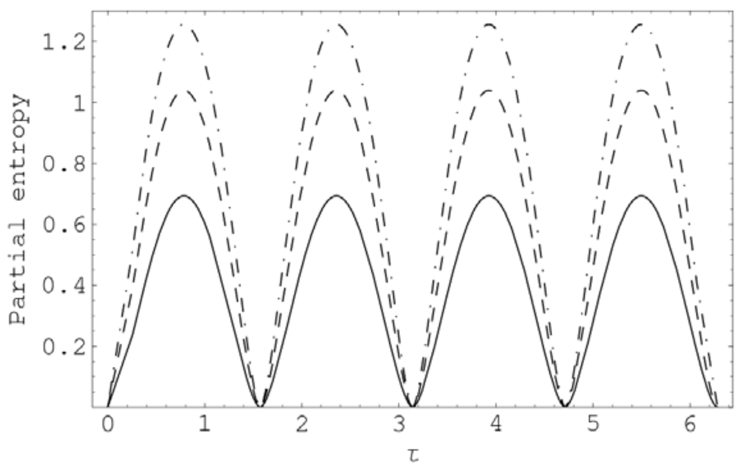

Figure 2. Partial entropies for k = 1 (continuous line), k = 2 (dashed line) and k = 3 (dot-dashed line).

It is very important to determine if the entanglement is maximal or not. For our system with k = 0 is a trivial case, because the wave function is separable anytime. For k = 1 the maximal entanglement occurs when τ is multiple integer of $\pi/4$ which correspond to the typical two level Bell states. In fact, these values coincide

[2] According to [10, 26] the conditional entropy indicates the entropy of one subsystem after measuring the other, and it is defined as

$$S(A|B) = S_{AB} - S_B,$$

while the mutual entropy indicates the entropy shared between the two subsystems, and it is given by

$$S(A:B) = S_A + S_B - S_{AB}.$$

with the maximal points in the partial entropy function (see figure 2). For higher values of k we have a pure multilevel bipartite system and then the only reliable criterion for entanglement is that conditional entropy $S(a|b)$ must be negative [10, 26, 28]. And this holds for our system according to (21), excepted when τ is a multiple integer of $\pi/2$ (for these times the system is separable).

Conclusions

In this work, we have examined the temporary evolution of the noncommutative two dimensional harmonic oscillator through the evolution operator acting over the pure number state $|0,k\rangle$. Firstly, the population inversion has not collapses and revivals as it is shown in figure 1. In figure 2 the entanglement is presented in terms of the partial entropies for different values of an initial excitation number k. The entanglement naturally arises from the third term appearing in the Hamiltonian (3). This term is due to the noncommutativity of spatial coordinates. Furthermore, the maximal entanglement is reached when the bipartite state takes a Bell state form, with τ multiple integer of $\tau/4$ for $k=1$. In this way, an experimental measurement procedure, based on thermodynamics quantities, can be established in order to find out if the spatial coordinates are commutative or not. Many noncommutative quantization proposals have been appeared for the two dimensional harmonic oscillator. Though it is very important to remark that the one used in this paper led us to a Hamiltonian possessing a $SU(2)$ symmetry and the time evolution operator was constructed in terms of this algebra. However, a different quantization of the Hamiltonian (1) could give another symmetry, nevertheless the same technique showed here could be used.

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